

K-theory of hermitian symmetric spaces and root lattices

Dennis Bohle, Wend Werner
 Fachbereich Mathematik und Informatik
 Westfälische Wilhelms-Universität
 Einsteinstraße 62
 48149 Münster

dennis.bohle@math.uni-muenster.de,
 wwerner@math.uni-muenster.de

Abstract

This is a companion to a recent investigation of K-theoretical invariants for symmetric spaces. We introduce a new class of cycles in K-groups, which are connected to elements of an underlying root lattice. This will be needed for a K-theoretical classification of inductive limits.

1 Introduction

In a parallel investigation [BW12], inductive limits of bounded symmetric domains are classified through a variant of K-theory. The latter has recently been shown to yield a homological method for the classification of the classical bounded symmetric domains [BW11a].

In finite dimensions, the Cartan/Serre approach requires a Hilbert space structure on the vector spaces containing the root system of the underlying Lie algebra. Also in [BW11a] roots (or, more precisely, grids) have been used to mark K-groups in order to achieve a complete invariant. Using root systems is problematic if treating inductive limits: The Hilbertian structure, along the limit, disappears completely, and the connecting morphisms may be of higher multiplicity, which makes it difficult to keep track of the underlying root systems.

In the sequel we try to promote the idea to use a different invariant, a marked subset of the group K_0 , which is closely linked to root lattices. As can be seen in [BW12], this new ingredient is very helpful in infinite dimensions.

We will provide some background information in the next section, define and motivate the new K-theoretical invariant in the third, and then set out for a calculation in the final section.

2 Partial isometries, Grids and Roots

Important for the following is the Kaup-Koecher-Loos approach to symmetric complex domains. It features a threefold product, defined on the Banach space E surrounding the domain U . This product turns E into an algebraic object, a JB*-triple system, more general than a C*-algebra.

Definition 2.1. *A Banach space Z together with a trilinear, continuous and w.r.t. the outer variables symmetric mapping $\{\cdot, \cdot, \cdot\} : Z^3 \rightarrow Z$ is called a JB*-triple, iff*

- (a) $\|\{x, x, x\}\| = \|x\|^3$ for all $x \in Z$,
- (b) With the operator $x \square y$ defined on Z by $(x \square y)(z) = \{x, y, z\}$, $ix \square y$ is a derivation,
- (c) $x \square x$ has non-negative spectrum, and $\exp(it(x \square x))$ is a 1-parameter group of isometries.

It turns out that complex bounded domains in Banach spaces are, up to biholomorphic bijections, just the open unit balls of JB*-triple systems [Kau77]. In finite dimension, a Wedderburn type result shows that JB*-triple systems can be decomposed into direct sums of *Cartan factors*, which will be detailed below.

The structure of a JB*-triple system is intimately connected to the Lie algebra of all complete holomorphic vector fields defined on its open unit ball. (Recall that a *complete vector field* is supposed to have a flow defined for all times.) A variant of this relationship is the Tits-Kantor-Koecher Algebra of Z , which is given as the Lie (sub)algebra of quadratic vector fields

$$\mathfrak{tkk}(Z) = \left\{ z \mapsto a + \sum_i \{v_i, w_i, z\} + \{z, b, z\} \mid a, b, v_i, w_i \in Z \right\}.$$

It is 3-graded, $\mathfrak{tkk}(Z) = \mathfrak{tkk}(Z)_{-1} \oplus \mathfrak{tkk}(Z)_0 \oplus \mathfrak{tkk}(Z)_1$, where $\mathfrak{tkk}(Z)_0$ is the Lie subalgebra of linear vector fields, and

$$\mathfrak{tkk}(Z)_\pm = \{ z \mapsto a \pm \{z, a, z\} \mid a \in Z \}.$$

Root systems of $\mathfrak{tkk}(Z)$ can be recovered from the JB*-triple structure as they leave an imprint on the set of *tripotents*, i.e. elements $z \in Z$ with $\{z, z, z\} = z$. Central to this relationship is the concept of a *grid* or, more generally of a *cog*, both sets of tripotents satisfying a number of relations. This connection has been thoroughly explored by Neher [Neh91, Neh90] who has shown how to use grids for a classification of finite dimensional bounded symmetric domains based on generators and relations.

We recall the definition of a root system. Let X be a real, finite-dimensional vector space with scalar product $\langle \cdot, \cdot \rangle$. A subset $R \subseteq X$ is called a *root system* iff

- (a) R is finite, generates X and does not contain 0.
 - (b) For every root $\alpha \in R$ we have $s_\alpha(R) = R$, where $s_\alpha(x) = x - 2\frac{\langle x, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha$ is the reflection in α .
 - (c) $(\alpha, \beta) := 2\frac{\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} \in \mathbb{Z}$ for all $\alpha, \beta \in R$.
 - (d) For every $\alpha \in R$, we have $\mathbb{R}\alpha \cap R = \{\pm\alpha\}$.
- A root system $R \neq \emptyset$ is called *irreducible* if R cannot be decomposed into two orthogonal non-empty subsets. The root system R is *3-graded* if additionally there exist $R_{-1}, R_0, R_1 \subseteq R$ with
- (d) $R = R_1 \dot{\cup} R_0 \dot{\cup} R_{-1}$ (disjoint union).
 - (e) $R_{-1} = -R_1$.
 - (f) $R_0 = \{\alpha - \beta; \alpha, \beta \in R_1, \alpha \neq \beta, \langle \alpha, \beta \rangle = 0\}$.
 - (g) If $\alpha, \beta \in R_1$ then $\alpha + \beta \notin R$.
 - (h) If $\alpha \in R_0, \beta \in R_1$ and $\alpha + \beta \in R$ then $\alpha + \beta \in R_1$.

Due to the conditions (e) and (f) the grading of a root system is completely determined by its (R_1) part.

It can be shown that for each (closed) cog in Z there is a 3-graded root system R for $\mathfrak{tff}(Z)$ as well as a bijection respecting the relations among cog and, respectively, root elements. Irreducible 3-graded root systems correspond to *connected grids*.

The relation between roots and grids is as follows: Identify Z with the 1-part of its Kantor-Koecher-Tits algebra $\mathfrak{tff}(Z)$. The root system R of $\mathfrak{tff}_0(Z)$ with respect to a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{tff}_0(V)$ is then naturally 3-graded, and the spaces $\mathfrak{tff}_i(Z)$, $i = \pm 1, 0$, are spanned by the root spaces $\mathfrak{tff}(Z)_\alpha$ with $\alpha \in R_i$. Moreover, relative to a properly chosen $\mathfrak{h} \subseteq \mathfrak{tff}_0(V)$, the root spaces $\mathfrak{tff}(Z)_\alpha$, $\alpha \in R_i$, are exactly the one dimensional subspaces spanned by the elements of the grid.

Every connected grid \mathcal{G} is associated to one of the following standard grids. We include a list of the 3-graded root systems corresponding to the standard grids and thus to the classical Cartan factors. For details see [Neh87], [Neh96, §3].

Cartan factor $\mathbf{C}_{n,m}^1$, of type I This is the space of complex $n \times m$ -matrices $\mathbb{M}_{n,m}(\mathbb{C})$, the *rectangular factor*, with triple product $1/2(AB^*C + CB^*A)$. Its (reduced) grid $\mathcal{R}(n, m)$, $n, m \geq 1$ is called rectangular and is given by the usual matrix units $\{E_{i,j} \mid 1 \leq i \leq m, 1 \leq j \leq n\}$. The corresponding root system R lies in the subspace

$$X = \left\{ \sum_{1 \leq k \leq m+n} s_k e_k \in \ell^2(m+n) \mid \sum_{1 \leq k \leq m+n} s_k = 0 \right\}$$

of the (real) Hilbert space $\ell^2(m+n)$. (Here, as in the following, e_i are elements of an orthonormal basis.) It is given by

$$R = \mathcal{A}_{m+n-1} = \{e_k - e_l \mid 1 \leq k, l \leq m+n, k \neq l\}$$

with grading induced by

$$R_1 = \{e_i - e_j \mid 1 \leq i \leq m, 1 \leq j \leq n\}.$$

Cartan factor C_n^2 , of type II A space of skew-symmetric, complex $n \times n$ -matrices, $n \geq 4$, called *symplectic factor*. The standard example of the symplectic grid $\mathcal{S}(n)$, $n \geq 5$, is $\{E_{i,j} - E_{j,i} \mid 1 \leq i, j \leq n, i < j\}$, the corresponding 3-graded root system is a subset of $\ell^2(n)$, and is given by

$$R = \mathcal{D}_n = \{\pm e_i \pm e_j \mid 1 \leq i, j \leq n, i \neq j\},$$

with grading given by

$$R_1 = \{e_i + e_j \mid 1 \leq i, j \leq n, i \neq j\}.$$

Cartan factor C_n^3 , of type III This is the *hermitian factor*, consisting of symmetric complex $n \times n$ -matrices, $n \geq 2$. For the hermitian grid $\mathcal{H}(n)$, $n \geq 2$, the standard example is $\{E_{i,j} + E_{j,i} \mid 1 \leq i, j \leq n, i \neq j\} \cup \{E_{i,i} \mid 1 \leq i \leq n\}$ the associated root system is

$$R = \mathcal{C}_n = \{\pm 2e_i \mid 1 \leq i \leq n\} \cup \{\pm e_i \pm e_j \mid 1 \leq i, j \leq n\}$$

with 1-part

$$R_1 = \{e_i + e_j \mid 1 \leq i, j \leq n\},$$

again as a subset of $\ell^2(n)$.

Cartan factor C_n^4 , of type IV The $n+1$ -dimensional *spin factor*, $n \geq 2$ is the closed linear span of selfadjoint matrices $1, s_1, \dots, s_n$, satisfying

$$s_i s_j + s_j s_i = 2\delta_{ij} 1$$

for all $i, j \in \{1, \dots, n\}$. Such a set of matrices is called a *spin system*. The standard way of constructing one is to start with the Pauli matrices

$$\sigma_1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \sigma_2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_3 := \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

For a matrix σ , write σ^k to denote its k -fold tensor product. Then

$$\begin{aligned}
s_0 &= id^n, \\
s_1 &= \sigma_1 \otimes id^{n-1}, \\
s_2 &= \sigma_2 \otimes id^{n-1}, \\
s_3 &= \sigma_3 \otimes \sigma_1 \otimes id^{n-2}, \\
s_4 &= \sigma_3 \otimes \sigma_2 \otimes id^{n-2}, \\
s_{2l+1} &= \sigma_3^l \otimes \sigma_1 \otimes id^{n-l-1}, \\
s_{2l+2} &= \sigma_3^l \otimes \sigma_2 \otimes id^{n-l-1},
\end{aligned}$$

$1 \leq l \leq n-1$, becomes a spin system which generates the odd dimensional spin factor. If we drop the last idempotent s_{2n} we get a spin system which generates an even dimensional spin factor. The spin grid $\mathcal{Sp}(n) = \{u_i, \tilde{u}_i \mid i = 1, \dots, n\}$, $n \geq 1$ is obtained in the following way. (We follow here an idea from [Boh11], which in turn is based on [NR03].) If n is odd (i.e. the corresponding spin factor is of even dimension) the elements of the $\mathcal{Sp}(n)$ are given by

$$\begin{aligned}
u_1 &= \frac{1}{2}(id - s_1) \quad \tilde{u}_1 = -\frac{1}{2}(id + s_1) \quad \text{and} \\
u_{k+1} &= \frac{1}{2}(s_{2k} + is_{2k+1}), \quad \tilde{u}_{k+1} = \frac{1}{2}(s_{2k} - is_{2k+1}) \quad \text{for } k = 1, \dots, \frac{1}{2}(n-1),
\end{aligned}$$

whereas in the case that n is even, $\mathcal{Sp}(n) = \{u_i, \tilde{u}_i \mid i = 1, \dots, n\} \cup \{u_0\}$ with $u_0 = s_n$. The 3-graded root system associated to $\mathcal{Sp}(n)$, n even, is a root system of type \mathcal{D}_{n+1} in the vector space $\ell^2(n) \oplus \mathbb{R}e_\infty$,

$$R = \mathcal{D}_{n+1} = \{\pm e_i \pm e_j \mid i, j = 1, \dots, n, \infty, i \neq j\},$$

where the grading comes from

$$R_1 = \{e_\infty \pm e_i \mid i = 1, \dots, n\}.$$

For a spin grid $\mathcal{Sp}(n)$ with $|\mathcal{Sp}(n)|$ odd the associated root system R is of type \mathcal{B}_{n+1} ,

$$R = \mathcal{B}_{n+1} = \{\pm e_i \pm e_j \mid i, j = 1, \dots, n, \infty, i \neq j\} \cup \{\pm e_i \mid i = 1, \dots, n, \infty\},$$

3-graded with 1-part

$$R_1 = \{e_\infty \pm e_i \mid i = 1, \dots, n\} \cup \{e_\infty\}.$$

Exceptional factors There are two exceptional factors in dimensions 16 and 27. And the two exceptional grids

(V) Bi-Caley grid.

(VI) Albert grid.

Unfortunately, they do not play a role here.

These factors are mutually non-isomorphic with the exception of $\mathbb{M}_{2,2}$, the symplectic factor for $n = 4$, and the hermitian factor for $n = 2$, which are spin factors, of dimensions 4, 6 and 3, respectively.

3 The K-JB* invariant

Recall the following definitions from [BW11a]. For unexplained notation on ternary rings of operators (TROs, for short), we refer the reader to [BLM04].

K-groups for JB*-triples are obtained in two steps. The first defines K-groups for TROs: For a TRO T , $K_*(T)$ is the K-group of its left C*-algebra. As each TRO-morphism ψ yields C*-morphisms $\mathcal{L}(\psi)$ and $\mathcal{R}(\psi)$ between both, left and right C*-algebras, we may define $K_*(\psi)$ in a functorial way. Note that for TROs, we stick to writing $K_*(\cdot)$ for their K-groups, as these groups coincide with their C*-counterparts should they themselves be C*-algebras. The second step involves the enveloping TRO of a JB*-triple [BW11b, BFT12]. To each JB*-triple corresponds a *universal (enveloping) TRO* $T^*(Z)$, as well as a canonical embedding $\rho_Z : Z \rightarrow T^*(Z)$ such that the image $\rho_Z(Z)$ generates $T^*(Z)$ as a TRO. Furthermore, each JB*-morphism uniquely lifts to a TRO-morphism between the respective universal TROs, and the emerging functor, denoted τ , has all properties needed for the ensuing K-theory. For a JB*-triple Z and a JB*-morphism $\phi : Z \rightarrow W$, we define

$$K_*^{\text{JB}^*}(Z) = K_*^{\text{TRO}}(\tau(Z)),$$

as well as

$$K_*^{\text{JB}^*}(\phi) = K_*^{\text{TRO}}(\tau(\phi))$$

This K-functor has the usual properties that one would expect from it, except stability, which already has a bad start, as it is not possible, in general, to equip the space of matrices with entries from a JB*-triple with the structure of a JB*-triple in a natural way.

A complete isomorphism invariant for finite dimensional JB*-triples is obtained by equipping K-groups with further structure, consisting of some distinguished subsets of $K_0(Z)$. The first is the semigroup $K_0(Z)^+$, consisting of all classes of projections themselves, that is, the set obtained before, in the final step, the Grothendieck construction is applied in order to produce $K_0(Z)$. The next is the scale of the left C*-algebra of $T^*(Z)$, i.e. classes of projections in $\mathcal{L}(T^*(Z))$. We access the scale of the right C*-algebra of $T^*(Z)$ with the aid of a canonical mapping which (at least in case of separable C*-algebras) is defined in the following way. It can be shown that the canonical mappings $\iota_{\mathcal{L}} : \mathcal{L}(T) \rightarrow \mathbb{L}(T)$ and $\iota_{\mathcal{R}} : \mathcal{R}(T) \rightarrow \mathbb{L}(T)$ induce

isomorphisms

$$K_0(\iota_{\mathcal{R}}) : K_0(T) \rightarrow K_0(\mathbb{L}(T)) \quad \text{and} \quad K_0(\iota_{\mathcal{R}}) : K_0(\mathcal{R}(T)) \rightarrow K_0(\mathbb{L}(T)).$$

We then let the image of $K_0(\iota_{\mathcal{R}})K_0(\iota_{\mathcal{R}})^{-1}$ represent the scale of $\mathcal{R}(T^*(Z))$ in $K_0(\mathcal{R}(T))$.

So far, the data we have assembled wouldn't suffice for a complete isomorphism invariant. In [BW11a], K-classes generated by grid elements were shown to provide just the right amount of additional information missing before. These classes, however, show an inappropriate behavior when dealing with morphisms other than automorphisms. This defect may be overcome by considering the lattice generated by grid elements. If, however, we follow along the lines of the correspondence between roots and grid elements, a more practicable approach comes into view. Consider an element

$$\beta = \sum k_i \alpha_i, \quad k_i \in \mathbb{Z}.$$

of the root lattice of $\mathfrak{tff}(Z)$, generated by a root system for a properly chosen Cartan-subalgebra $\mathfrak{h} \subseteq \mathfrak{tff}(Z)_0$. Then all elements $\sum \lambda_i g_i$, $\lambda_i \in \mathbb{C}$ and $g_i \in \mathfrak{tff}(Z)_{\alpha_i}$ belong to

$$\mathfrak{tff}(Z)_{\beta} = \{ z \in Z \mid [h, z] = \beta(h)z \text{ for all } h \in \mathfrak{h} \}.$$

But, as all tripotents of Z can be written in such a way, we use

$$\Delta(Z) := \{ [\rho_Z(g)\rho_Z(g)^*] \in K_0^{JB^*}(Z) \mid g \in \text{Tri}(Z) \}$$

as an additional ingredient for the isomorphism invariant. This set carries as much information as the root lattice of $\mathfrak{tff}(Z)$ might provide on the level of K-groups. Summing up,

Definition 3.1. *Let Z be a JB^* -triple system. The K - JB^* invariant of Z is the tuple*

$$\mathcal{KJB}^*(Z) := (K_0^{JB^*}(Z), K_0^{JB^*}(Z)_+, \Sigma_{\mathcal{L}}^{JB^*}(Z), \Sigma_{\mathcal{R}}^{JB^*}(Z), \Delta(Z)),$$

where $K_0^{JB^*}(Z)_+ := K_0(T^*(Z))_+$, $\Sigma_{\mathcal{L}}^{JB^*}(Z)$ and $\Sigma_{\mathcal{R}}^{JB^*}(Z)$ are the left and right scale of the TRO $T^*(Z)$ and $\Delta(Z)$ is the set of equivalence classes coming from the set of tripotents in Z .

4 Calculation

Since in finite dimensional C^* -algebras projections are equivalent iff they are unitarily equivalent, the following lemma follows from [BW12, Lemma 3.1].

Lemma 4.1. *Let u, v be tripotents in a finite dimensional JB^* -triple Z . Then the following are equivalent.*

- (i) u and v yield the same class in $K_0^{JB^*}(Z)$
- (ii) There is a unitary $U \in \mathcal{L}(T^*(Z))$ such that $Uv = u$.
- (iii) There is an automorphism of Z mapping u onto v .

The TRO $T^*(Z_1)$ is the finite sum of rectangular matrix algebras $T^*(Z_1) \simeq \bigoplus_{i=1}^p \mathbb{M}_{n_i, m_i}$ [Smi00]. This structure is perfectly reflected by the double-scaled ordered K_0 -group of $T^*(Z_1)$.

Definition 4.2. *Let $\varphi : K_0^{JB^*}(Z_1) \rightarrow K_0^{JB^*}(Z_2)$ be a map. We say that φ is a K - JB^* morphism if φ is a group morphism with*

$$\begin{aligned} \varphi(K_0^{JB^*}(Z)_+) &\subseteq K_0^{JB^*}(W)_+, & \varphi(\Sigma_{\mathcal{L}}^{JB^*}(Z_1)) &\subseteq \Sigma_{\mathcal{L}}^{JB^*}(Z_2), \\ \varphi(\Sigma_{\mathcal{R}}^{JB^*}(Z_1)) &\subseteq \Sigma_{\mathcal{R}}^{JB^*}(Z_2), & \text{and } \varphi(\Delta(Z_1)) &\subseteq \Delta(Z_2). \end{aligned}$$

The proof of the following result is obvious (and follows along the lines of [BW11a, Proposition 4.5])

Proposition 4.3. *Let Z_1 and Z_2 be finite dimensional JB^* -triple systems, then there exists a \mathcal{KJB}^* -isomorphism of K_0 -groups*

$$\mathcal{KJB}^*(Z_1 \oplus Z_2) \simeq \mathcal{KJB}^*(Z_1) \oplus \mathcal{KJB}^*(Z_2).$$

Proposition 4.4. *Let Z be a JB^* -triple system*

(a) *Suppose Z is the finite-dimensional type I Cartan factor $Z = C_{n,m}^1$.*

(i) *If Z is isometric to a finite-dimensional Hilbert space, i.e. $Z = C_{1,n}^1$, $n \in \mathbb{N}$, then $\mathcal{KJB}^*(C_{1,n}^1)$ is given by*

$$\begin{aligned} K_0^{JB^*}(C_{1,n}^1) &= \mathbb{Z}^n, \\ K_0^{JB^*}(C_{1,n}^1)_+ &= \mathbb{N}_0^n, \\ \Sigma_{\mathcal{L}}^{JB^*}(C_{1,n}^1) &= \prod_{k=1}^n \left\{ 1, \dots, \binom{n}{k} \right\}, \\ \Sigma_{\mathcal{R}}^{JB^*}(C_{1,n}^1) &= \prod_{k=1}^n \left\{ 1, \dots, \binom{n}{k-1} \right\} \text{ and} \\ \Delta(C_{1,n}^1) &= \left\{ \left(\binom{n-1}{0}, \binom{n-1}{1}, \dots, \binom{n-1}{n-1} \right) \right\}. \end{aligned}$$

(ii) *If $n, m \geq 2$, then $\mathcal{KJB}^*(C_{n,m}^1)$ is given by*

$$\begin{aligned} \mathcal{KJB}^*(C_{n,m}^1) &= (\mathbb{Z}^2, \mathbb{N}_0^2, \{1, \dots, n\} \times \{1, \dots, m\}, \\ &\quad \{1, \dots, m\} \times \{1, \dots, n\}, \\ &\quad \{1, \dots, n\} \times \{1, \dots, m\}). \end{aligned}$$

(b) Let Z be isometric to a Cartan factor of type II with $\dim Z \geq 10$. Then

$$\mathcal{KJB}^*(C_n^2) = (\mathbb{Z}, \mathbb{N}_0, \{1, \dots, n\}, \{1, \dots, n\}, \{2, 4, \dots, k\}),$$

where k is the greatest even integer less or equal to n .

(c) If Z is JB^* -triple isomorphic to the finite-dimensional Cartan factor C_n^3 , then

$$\mathcal{KJB}^*(C_n^3) = (\mathbb{Z}, \mathbb{N}_0, \{1, \dots, n\}, \{1, \dots, n\}, \{1, \dots, n\}).$$

(d) Let Z be a finite-dimensional spin factor with $\dim Z = k + 1$.

(i) If Z is of even dimension, i.e. $k = 2n - 1$, $n \geq 2$, then the K - JB^* invariant of Z is given by

$$\begin{aligned} K_0^{JB^*}(Z) &= \mathbb{Z}^2, \\ K_0^{JB^*}(Z)_+ &= \mathbb{N}_0^2, \\ \Sigma_{\mathcal{L}}^{JB^*}(Z) &= \{1, \dots, 2^{n-1}\}^2, \\ \Sigma_{\mathcal{R}}^{JB^*}(Z) &= \{1, \dots, 2^{n-1}\}^2, \\ \Delta(Z) &= \{(2^{n-2}, 2^{n-2}), (2^{n-1}, 2^{n-1})\}. \end{aligned}$$

(ii) If Z is of odd dimensions, i.e. $k = 2n$, $n \geq 2$, then $\mathcal{KJB}^*(Z)$ has the components

$$\begin{aligned} K_0^{JB^*}(Z) &= \mathbb{Z}, \\ K_0^{JB^*}(Z)_+ &= \mathbb{N}_0, \\ \Sigma_{\mathcal{L}}^{JB^*}(Z) &= \{1, \dots, 2^n\}, \\ \Sigma_{\mathcal{R}}^{JB^*}(Z) &= \{1, \dots, 2^n\}, \\ \Delta(Z) &= \{2^{n-1}, 2^n\}. \end{aligned}$$

Proof. All the K_0 -groups, their positive cones and their scales were computed in [BW11a], so we have to compute the new invariant $\Delta(Z)$ for all Cartan factors Z . In light of Lemma 4.1, the result for the (non-Hilbertian) rectangular, symplectic and hermitian cases are rather obvious. We prove the remaining cases, Hilbert spaces and spin factors.

The tripotents e in a Hilbert space, when the latter is viewed as a Cartan factor of type I (and rank 1), are the norm-one elements. Thus, whenever e, f are tripotents, there exists an automorphism U , i.e. a unitary map, mapping e to f . Hence, $\Delta(Z)$ coincides with the classes generated by the grid elements which in light of [BW11a, Proposition 5.2.] completes the proof of (a)(i).

If Z is a spin factor of dimension n , tripotents belong to one of two distinct classes, which can be seen as follows. Recall that on a spin factor

$Z = \mathbb{C}^n$ there is always a scalar product as well as an involution $a \mapsto \bar{a}$ so that

$$\{a, b, c\} = \langle a, b \rangle c + \langle c, b \rangle a - \langle a, \bar{c} \rangle \bar{b}$$

for $a, b, c \in \mathbb{C}^n$. Thus $e \in Z$ is a tripotent iff

$$e = \{e, e, e\} = 2\langle e, e \rangle e - \langle e, \bar{e} \rangle \bar{e},$$

and we have the following possibilities:

(a) $\langle e, \bar{e} \rangle = 0$ and $\langle e, e \rangle = \frac{1}{2}$.

(b) $\langle e, \bar{e} \rangle \neq 0$, $e = \mu r$ with $\mu \in \mathbb{C}$, $|\mu| = 1$ and $r \in \mathbb{C}^n$, $\bar{r} = r$ and $\langle r, r \rangle = 1$.

Following the notation of [Fri05, Chapter 3], the tripotents in (a) are called minimal and those in (b) maximal. We will show that that two minimal tripotents yield equivalent elements in $K_0(Z)$ and likewise for two maximal tripotents.

Let e and f be maximal tripotents. Thus $e = \mu_1 r_1$ and $f = \mu_2 r_2$ as above. It is well known [Har74, Corollary 5] that a linear map Φ is an automorphism of Z iff $\Phi = \lambda \Phi_0$, where λ is a complex scalar of modulus 1 and Φ_0 is unitary (for the scalar product $\langle \cdot, \cdot \rangle$) such that $\Phi_0(\bar{x}) = \overline{\Phi_0(x)}$ for all $x \in Z$. Let Φ_{00} be an orthogonal mapping defined on the self-adjoint part of Z , mapping r_1 to r_2 . Denote by Φ_0 its complexification (which then is an automorphism of the spin factor). If we define $\Phi := \bar{\mu}_1 \mu_2 \Phi_0$, then Φ is an automorphism of Z mapping e to f and the corresponding projections are equivalent in $K_0(Z)$.

It follows from [Fri05, 3.1.4] that if we decompose a minimal tripoten v as

$$v = x + iy, \quad x, y \text{ self-adjoint},$$

then

$$|x| = |y| = \frac{1}{2} \quad \text{and} \quad \langle x, y \rangle = 0.$$

Let now $e = x_1 + iy_1$ and $f = x_2 + iy_2$ be minimal tripotents decomposed in this way. Pick e_1^i, \dots, e_{n-2}^i , such that $2x_i, 2y_i, e_1^i, \dots, e_{n-2}^i$ is an orthonormal basis of the self-adjoint part of Z , for $i = 1, 2$. Let Φ_{00} be the linear mapping with $\Phi_{00}(x_1) = x_2$, $\Phi_{00}(y_1) = y_2$ and $\Phi_{00}(e_k^1) = e_k^2$ for $k = 1, \dots, n-2$. The complexification Φ_0 of Φ_{00} is obviously an automorphism of Z that maps e to f and therefore the elements in $K_0(Z)$ that correspond to e and f are equivalent.

Along the same lines (using once more Lemma 4.1, and Harris' characterization of the automorphisms), no maximal tripotent generates the same class as a minimal one. Thus, for a spin factor Z , $\Delta(Z)$ has exactly 2 elements. One of them always belongs to the spin grid, whereas the other is in the class of the identity of Z (which in the evendimensional case is equivalent to a grid element as well). Details are in the proof of [BW11a, Proposition 5.1]. \square

We will show that for finite dimensional JC^* -triple system Z and W we can lift K_0 -group-homomorphisms that respect the order structure and the two scales to TRO-homomorphisms of the universal enveloping TRO of Z and W . The additional data implemented by $\Delta(Z)$ will be used afterwards to determine the JB^* -isomorphisms between Z and W .

Proposition 4.5. *Let Z and W be finite-dimensional JC^* -triple systems and $T^*(Z) = \bigoplus_{i=1}^p \mathbb{M}_{n_i, m_i}$ and $T^*(W) = \bigoplus_{j=1}^q \mathbb{M}_{k_j, l_j}$ their universal enveloping TROs. Suppose*

$$\begin{aligned} (K_0^{JB^*}(Z), K_0^{JB^*}(Z)_+, \Sigma_{\mathcal{L}}^{JB^*}(Z), \Sigma_{\mathcal{R}}^{JB^*}(Z)) = \\ = \left(\mathbb{Z}^p, \mathbb{N}_0^p, \prod_{i=1}^p \{0, \dots, n_i\}, \prod_{i=1}^p \{0, \dots, m_i\} \right) \end{aligned}$$

and

$$\begin{aligned} (K_0^{JB^*}(W), K_0^{JB^*}(W)_+, \Sigma_{\mathcal{L}}^{JB^*}(W), \Sigma_{\mathcal{R}}^{JB^*}(W)) = \\ = \left(\mathbb{Z}^q, \mathbb{N}_0^q, \prod_{j=1}^q \{0, \dots, k_j\}, \prod_{j=1}^q \{0, \dots, l_j\} \right). \end{aligned}$$

Let $\alpha : K_0^{JB^*}(Z) \rightarrow K_0^{JB^*}(W)$ be a homomorphism such that

$$\alpha(K_0^{JB^*}(Z)_+, \Sigma_{\mathcal{L}}^{JB^*}(Z), \Sigma_{\mathcal{R}}^{JB^*}(Z)) \subseteq (K_0^{JB^*}(W)_+, \Sigma_{\mathcal{L}}^{JB^*}(W), \Sigma_{\mathcal{R}}^{JB^*}(W)).$$

Then there exists a TRO-homomorphism $\varphi : T^*(Z) \rightarrow T^*(W)$ with $K_0(\varphi) = \alpha$.

Proof. We will use the fact that $\alpha : \mathbb{Z}^p \rightarrow \mathbb{Z}^q$ can be represented as a $q \times p$ -matrix $(a_{i,j})_{i,j}$ with entries $a_{i,j} \in \mathbb{N}_0$. Fix $x = (z_1, \dots, z_p) \in \Sigma_{\mathcal{L}}^{JB^*}(Z)$. Since $\alpha(\Sigma_{\mathcal{L}}^{JB^*}(Z)) \subseteq \Sigma_{\mathcal{L}}^{JB^*}(W)$,

$$\alpha(x) = \left(\sum a_{1,i} z_i, \dots, \sum a_{q,i} z_i \right) \leq (k_1, \dots, k_q),$$

so $\sum_{j=1}^p a_{i,j} z_j \leq k_i$ for all $i = 1, \dots, q$, and similarly, $\sum_{j=1}^p a_{i,j} z_j \leq l_i$ for all $i = 1, \dots, q$. To finish the proof, let φ be the direct sum $\varphi := \varphi_1 \oplus \dots \oplus \varphi_q$, where $\varphi_j : Z \rightarrow \mathbb{M}_{k_j, l_j}$ is defined by

$$\varphi_j(x_1 \oplus \dots \oplus x_p) := \text{diag}(\underbrace{x_1, \dots, x_1}_{a_{1,j} \text{ times}}, \dots, \underbrace{x_p, \dots, x_p}_{a_{p,j} \text{ times}}, 0, \dots, 0),$$

for $j = 1, \dots, q$. These TRO-homomorphisms are well-defined by the above inequalities, and $K_0(\varphi) = \alpha$. \square

Theorem 4.6. *Let Z_1 and Z_2 be finite-dimensional JC^* -triple systems. If $\sigma : K_0^{JB^*}(Z_1) \rightarrow K_0^{JB^*}(Z_2)$ is an isomorphism with $\sigma(\mathcal{KJB}^*(Z_1)) = \mathcal{KJB}^*(Z_2)$, then there exists a JB^* -isomorphism $\varphi : Z_1 \rightarrow Z_2$ such that $K_0^{JB^*}(\varphi) = \sigma$.*

Proof. We have

$$\sigma(K_0^{\text{JB}^*}(Z)_+, \Sigma_{\mathcal{L}}^{\text{JB}^*}(Z), \Sigma_{\mathcal{R}}^{\text{JB}^*}(Z)) = (K_0^{\text{JB}^*}(W)_+, \Sigma_{\mathcal{L}}^{\text{JB}^*}(W), \Sigma_{\mathcal{R}}^{\text{JB}^*}(W))$$

and so, σ is an isomorphism of the double-scaled ordered K_0 -groups of $T^*(Z_1)$ and $T^*(Z_2)$. Using the above classification result for TROs, Proposition 4.5, we can find a complete isometry $\varphi' : T^*(Z_1) \rightarrow T^*(Z_2)$ with $K_0(\varphi') = \sigma$.

We will now frequently make use of the additivity of the \mathcal{KJB}^* -invariant, Proposition 4.3. The structure of the TRO $T^*(Z_1) \simeq \bigoplus_{i=1}^p \mathbb{M}_{n_i, m_i}$ is obviously determined by the double-scaled ordered K_0 -group of $T^*(Z_1)$. An inspection of the list of \mathcal{KJB}^* -invariants shows that the information which is encoded in $\Delta(Z_1)$ permits to determine the Cartan-type of each summand, and we can recover the image $\rho_{Z_1}(Z_1) \subseteq T^*(Z_1)$ up to (ternary) unitary equivalence (i.e an inner TRO-automorphism). The same works for $\rho_{Z_2}(Z_2) \subseteq T^*(Z_2)$, and it follows that Z_1 and Z_2 are isomorphic as JB^* -triples.

Let $\psi : T^*(Z_2) \rightarrow T^*(Z_2)$ be the TRO isomorphism mapping $\varphi'(\rho_{Z_1}(Z_1))$ to $\rho_{Z_2}(Z_2)$ (we construct ψ with the aid of the universal property of $T^*(Z_2)$). Since Z_2 is finite dimensional so is $T^*(Z_2)$ and thus ψ is automatically unitarily equivalent to the identity, and hence the identity on $K_0(T^*(Z_2))$. If we put

$$\varphi := \rho_{Z_2}^{-1} \circ \psi \circ \varphi' \circ \rho_{Z_1} : Z_1 \rightarrow Z_2,$$

where $\rho_{Z_2}^{-1} : \rho_{Z_2}(Z_2) \rightarrow Z_2$ is the inverse of ρ_{Z_2} restricted to its image, then φ is a JB^* -isomorphism with $K_0^{\text{JB}^*}(\varphi) = \sigma$. \square

Actually, a closer look at morphisms between Cartan factors reveals the following result[BW12]

Theorem 4.7. *Let Z_1 and Z_2 be finite-dimensional JC^* -triple systems having summands of type I, II, or III only. If $\sigma : K_0^{\text{JB}^*}(Z_1) \rightarrow K_0^{\text{JB}^*}(Z_2)$ is a homomorphism with $\sigma(\mathcal{KJB}^*(Z_1)) \subseteq \mathcal{KJB}^*(Z_2)$, then there exists a JB^* -homomorphism $\varphi : Z_1 \rightarrow Z_2$ such that $K_0^{\text{JB}^*}(\varphi) = \sigma$.*

References

- [BFT12] Leslie J. Bunce, Brian Feely, and Richard Timoney. Operator space structure of JC^* -triples and TROs, I. *Math. Z.*, 270(3-4):961–982, 2012.
- [BLM04] David P. Blecher and Christian Le Merdy. *Operator algebras and their modules—an operator space approach*, volume 30 of *London Mathematical Society Monographs. New Series*. The Clarendon Press Oxford University Press, Oxford, 2004. Oxford Science Publications.

- [Boh11] Dennis Bohle. K-theory for ternary structures. *PhD Thesis*, 2011.
- [BW11a] Dennis Bohle and Wend Werner. A K-theoretic approach to the classification of symmetric spaces. *Preprint*, 2011.
- [BW11b] Dennis Bohle and Wend Werner. The universal enveloping TRO of a JB^* -triple system. *Proc. Edinburgh Math. Soc. (to appear)*, 2011.
- [BW12] Dennis Bohle and Wend Werner. Inductive limits of finite dimensional hermitian symmetric spaces and k-theory. *Preprint*, 2012.
- [Fri05] Yaakov Friedman. *Physical applications of homogeneous balls*, volume 40 of *Progress in Mathematical Physics*. Birkhäuser Boston Inc., Boston, MA, 2005. With the assistance of Tzvi Scarr.
- [Har74] Lawrence A. Harris. Bounded symmetric homogeneous domains in infinite dimensional spaces. In *Proceedings on Infinite Dimensional Holomorphy (Internat. Conf., Univ. Kentucky, Lexington, Ky., 1973)*, pages 13–40. Lecture Notes in Math., Vol. 364, Berlin, 1974. Springer.
- [Kau77] Wilhelm Kaup. Algebraic characterization of symmetric complex Banach manifolds. *Math. Ann.*, 228(1):39–64, 1977.
- [Neh87] Erhard Neher. *Jordan triple systems by the grid approach*, volume 1280 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1987.
- [Neh90] Erhard Neher. Systèmes de racines 3-gradués. *C. R. Acad. Sci. Paris Sér. I Math.*, 310(10):687–690, 1990.
- [Neh91] Erhard Neher. 3-graded root systems and grids in Jordan triple systems. *J. Algebra*, 140(2):284–329, 1991.
- [Neh96] Erhard Neher. Lie algebras graded by 3-graded root systems and Jordan pairs covered by grids. *Amer. J. Math.*, 118(2):439–491, 1996.
- [NR03] Matthew Neal and Bernard Russo. Contractive projections and operator spaces. *Trans. Amer. Math. Soc.*, 355(6):2223–2262 (electronic), 2003.
- [Smi00] Roger R. Smith. Finite dimensional injective operator spaces. *Proc. Amer. Math. Soc.*, 128(11):3461–3462 (electronic), 2000.